

Wollo University

College of Natural Science

Department of Physics

Note on Mathematical Methods of
Physics II

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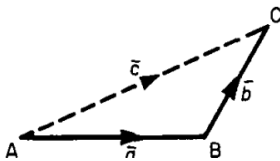
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Vectors and Matrices

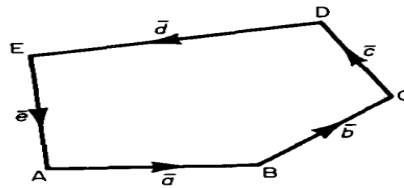
1.1 Algebra of Vectors

1. Addition of Vectors

To find the sum of two vectors $\vec{a}$ and $\vec{b}$ we draw them as a chain, starting the second where the first ends: the sum $\vec{c}$ is then given by the single vector joining the start of the first to the end of the second. Or, algebraically, the sum of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{c}$, such that $\vec{c} = (a_i + b_i)i + (a_j + b_j)j + (a_k + b_k)k$, where i, j, k are unit vectors.



What is the sum of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$?



2. Subtraction of Vectors

- The Subtraction of a vector $\vec{b}$ from another vector $\vec{a}$ is redefined as the sum of $\vec{a}$ and the negative of $\vec{b}$. $\vec{c} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$
- The negative of a vector $\vec{b}$ is a vector with the same magnitude and opposite direction.

3. Multiplication by a scalar

- If we multiply a vector $\vec{A}$ by a scalar α , the result is a vector $\vec{B} = \alpha\vec{A}$, which has magnitude $B = |\alpha|A$.
- The vector $\vec{B}$, is parallel to $\vec{A}$ and points in the same direction if $\alpha > 0$.
- For $\alpha < 0$, the vector $\vec{B}$ is parallel to $\vec{A}$ but points in the opposite direction. (anti-parallel)

4. Scalar multiplication

If $\vec{A}$ and $\vec{B}$ are two vectors, the scalar product of $\vec{A}$ and $\vec{B}$ is defined as $AB \cos \theta$. The dot product is meant as follows

$$\vec{A} \cdot \vec{B} = A \times \text{projection of B on A}$$

$$\text{or,} = B \times \text{projection of A on B}$$

5. Vector multiplication

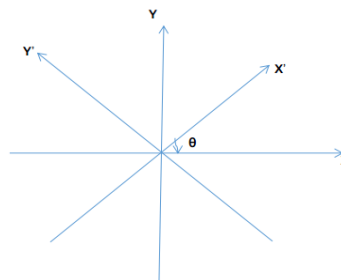
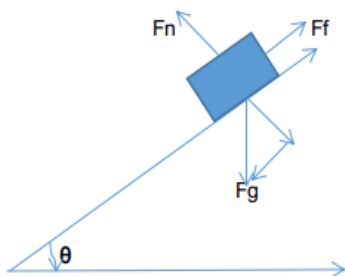
The vector product of $\vec{A}$ and $\vec{B}$ is written as $\vec{A} \times \vec{B}$ and is defined as a vector having a magnitude of $AB \sin \theta$. In matrix product represen-

tation

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_i & A_j & A_k \\ B_i & B_j & B_k \end{vmatrix}$$

1.2 Vector Space, Basis vectors and components

- A vector space is a set $\mathbf{V}$ of vectors such that with any two vectors $\vec{a}$ and $\vec{b}$ in $\mathbf{V}$ all their linear combinations $\alpha\vec{a} + \beta\vec{b}$ are elements of $\mathbf{V}$, and these vectors satisfy the laws of addition and multiplication of vectors.
- The maximum number of linearly independent vectors in $\mathbf{V}$ is called the dimension of $\mathbf{V}$.
- Here we assume the dimension to be finite.
- A linearly independent set in $\mathbf{V}$ consisting of a maximum possible number of vectors in $\mathbf{V}$ is called a basis for $\mathbf{V}$.



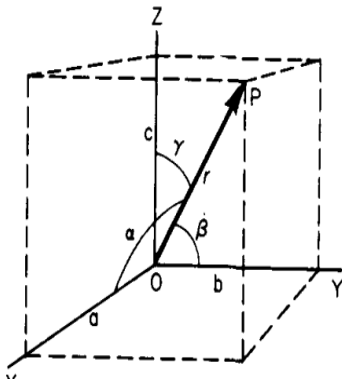
- $\vec{F}_g = -mgj$

- $\vec{F}_g = -mg \sin \theta \mathbf{i}' - mg \cos \theta \mathbf{j}'$
- The same force is expressed with different coordinates.
- Here $\mathbf{i}$ and $\mathbf{j}$, $\mathbf{i}'$ and $\mathbf{j}'$ are basis vectors.
- Any 3 non parallel vectors can be basis vectors in 3D.

Orthonormal basis: All basis vectors are perpendicular to each other and have unit length. Unit vectors are Orthonormal basis.

1.3 Direction Cosines

The direction of a vector in three dimensions is determined by the angles which the vector makes with the axes of the reference.



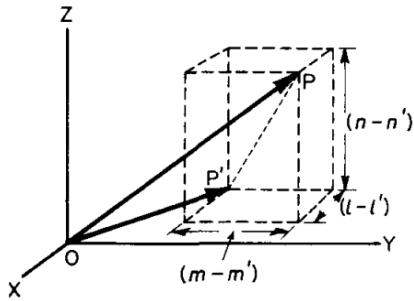
$$\begin{aligned}\cos \alpha &= \frac{a}{r} \\ \cos \beta &= \frac{b}{r} \\ \cos \gamma &= \frac{c}{r}\end{aligned}$$

Show that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

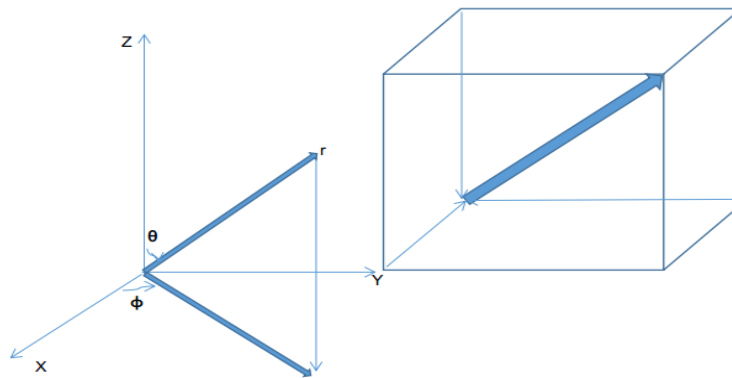
1.3.1 Angle between two vectors

If a vector $\vec{A}$ has direction cosines l , m , and n , and vector $\vec{B}$ has direction cosines l' , m' , and n' , then the angle between $\vec{A}$ and $\vec{B}$ is determined by

$$\cos \theta = ll' + mm' + nn'$$



1.3.2 Components of a vector in terms of unit vectors



Components of the vector $\vec{r}$ are the projection of $\vec{r}$ into the three directions.

$$\vec{r} = r_x \vec{i} + r_y \vec{j} + r_z \vec{k}$$

$$\text{where } r_x = r \sin \theta \cos \phi,$$

$$r_y = r \sin \theta \sin \phi$$

$$r_z = r \cos \theta$$

1.4 Matrix Algebra

Definition

- A matrix of order $m \times n$ is a rectangular array of numbers having m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- Each number a_{jk} in this matrix is called an element.

1. Addition of Matrices:

If matrices $A = (a_{jk})$ and $B = (b_{jk})$ have the same order, the sum of A and B is defined as $A + B = (a_{jk} + b_{jk})$

2. Subtraction of Matrices:

If matrices $A = (a_{jk})$ and $B = (b_{jk})$ have the same order, the difference of A and B is defined as $A - B = (a_{jk} - b_{jk})$

3. Multiplication of a Matrix by a Number:

If $A = (a_{jk})$ and λ is a number/scalar then multiplication of A by λ is defined as $\lambda A = \lambda a_{jk}$

4. Multiplication of Matrices:

If $A = (a_{jk})$ is an $m \times n$ and $B = (b_{jk})$ is an $n \times p$ matrices the product of A and B is defined as $AB = C = c_{jk}$ where

$$c_{jk} = \sum_{l=1}^n a_{jl} b_{lk}$$

The dimension of C is $m \times p$

5. Determinant:

The determinant of an $n \times n$ matrix can be expressed by the summation over any row or column,

$$D_n = \sum_{k=1}^n a_{jk} (-1)^{j+k} M_{jk}$$

or, $D_n = \sum_{j=1}^n a_{jk} (-1)^{j+k} M_{jk}$

where M_{jk} is the determinant produced by striking out row j and column k of the original determinant.

This technique is called expansion in minors.

Example Calculate the determinant of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Solution: Expanding across the top row, only one 3×3 matrix survives

$$D = (-1)^{1+2}a_{12}M_{12} = -1 \begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

Now expanding across the second row, we get $D = -1(1)(-1)^{2+3}M_{23} =$
 $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$

6. Inverse Matrix:

- For a square matrix A there will be a matrix B such that $AB = 1$.

Matrix B is called inverse of A. The formula for $(A)^{-1}$ is

$$(A)_{jk}^{-1} = \frac{(-1)^{j+k}M_{kj}}{\det(A)}$$

- Gauss - Jordan matrix inversion

Our strategy will be to write, side by side, a matrix A and a unit matrix of the same size, and to perform the same operations on each until A has been converted to a unit matrix, which means that the unit matrix will have been changed to A^{-1} .

Example Invert the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

We start with

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We multiply the rows as necessary to set to unity all elements of the first column of the left matrix:

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtracting the first row from the second and third rows, we obtain

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{11}{3} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

Then we divide the second row by $5/6$ and subtract $2/3$ times it from the first row and $1/3$ times it from the third row. The results for both matrices are

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & \frac{18}{5} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{3}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \\ -\frac{1}{5} & -\frac{1}{5} & 1 \end{pmatrix}$$

We divide the third row by $18/5$. Then as the last step, $1/5$ times the third row is subtracted from each of the first two rows. Our final pair

is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and, } A^{-1} = \begin{pmatrix} \frac{11}{18} & -\frac{7}{18} & -\frac{1}{18} \\ -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{18} \end{pmatrix}$$

1.5 Solving Systems of Linear Equations

1.5.1 Numerical Evaluation

1. GAUSS ELIMINATION

The Gauss technique is to use the first equation to eliminate the first unknown from the remaining equations. Then the (new) second equation is used to eliminate the second unknown from the last equation. In general, we work down through the set of equations, and then, with one unknown determined, we work back up to solve for each of the other unknowns in succession.

Solve the following system of equations.

$$3x + 2y + z = 11$$

$$2x + 3y + z = 13$$

$$x + y + 4z = 12$$

Solution

- We divide each row by its initial coefficient to get

$$\begin{aligned}x + \frac{2}{3}y + \frac{1}{3}z &= \frac{11}{3} \\x + \frac{3}{2}y + \frac{1}{2}z &= \frac{13}{2} \\x + y + 4z &= 12\end{aligned}$$

- We eliminate x from the second and third equations by subtracting the first equation from each of the others:

$$\begin{aligned}x + \frac{2}{3}y + \frac{1}{3}z &= \frac{11}{3} \\-\frac{5}{6}y + \frac{1}{6}z &= \frac{17}{6} \\\frac{1}{3}y + \frac{11}{3}z &= \frac{25}{3}\end{aligned}$$

- Then we divide the second and third rows by their initial coefficients:

$$\begin{aligned}x + \frac{2}{3}y + \frac{1}{3}z &= \frac{11}{3} \\y + \frac{1}{5}z &= \frac{17}{5} \\y + 11z &= 25\end{aligned}$$

- We use the new second equation to eliminate y from the third equation, which can then be solved for z:

$$\begin{aligned}x + \frac{2}{3}y + \frac{1}{3}z &= \frac{11}{3} \\y + \frac{1}{5}z &= \frac{17}{5} \\\frac{54}{5}z &= \frac{108}{5} \rightarrow z = 2\end{aligned}$$

- Going back to second and first equations we get, $y = 3$ and $x = 1$

Remark:

- In the above process the original determinant

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

was divided by 3 and then by 2, and was multiplied by $6/5$ and then by 3.

- So, to recover the original determinant we go the reverse process,

$$D = (3)(2)\left(\frac{5}{6}\right)\left(\frac{1}{3}\right) \begin{vmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & \frac{54}{5} \end{vmatrix} = \frac{5}{3} \frac{54}{5} = 18$$

Example Solve the following system of equations using matrix inversion.

$$3x + 2y + z = 11$$

$$2x + 3y + z = 13$$

$$x + y + 4z = 12$$

We rewrite the system as

$$AX = C$$

$$\rightarrow \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \\ 12 \end{pmatrix}$$

Then,

$$X = A^{-1}C$$

$$\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{11}{18} & -\frac{7}{18} & -\frac{1}{18} \\ -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{18} \end{pmatrix} \begin{pmatrix} 11 \\ 13 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

2. Cramer's Rule We are now ready to apply our knowledge of determinants to the solution of systems of linear equations. We see that the solution for x_i is $1/D$ times a numerator obtained by replacing the i^{th} column of D by the right-hand-side coefficients, a result that can be generalized to an arbitrary number n of simultaneous equations. This scheme for the solution of linear equation systems is known as Cramer's rule.

That is, if

$$a_1x_1 + a_2x_2 + a_3x_3 = h_1$$

$$b_1x_1 + b_2x_2 + b_3x_3 = h_2$$

$$c_1x_1 + c_2x_2 + c_3x_3 = h_3$$

then,

$$x_1 = \frac{1}{D} \begin{vmatrix} h_1 & a_2 & a_3 \\ h_2 & b_2 & b_3 \\ h_3 & c_2 & c_3 \end{vmatrix}$$

$$x_2 = \frac{1}{D} \begin{vmatrix} a_1 & h_1 & a_3 \\ b_1 & h_2 & b_3 \\ c_1 & h_3 & c_3 \end{vmatrix}$$

$$x_3 = \frac{1}{D} \begin{vmatrix} a_1 & a_2 & h_1 \\ b_1 & b_2 & h_2 \\ c_1 & c_2 & h_3 \end{vmatrix}$$

Example Solve the following system of equations using Cramer's rule

$$3x + 2y + z = 11$$

$$2x + 3y + z = 13$$

$$x + y + 4z = 12$$

We know the determinant of the matrix of coefficients is 18. So, we have

$$x = \frac{1}{18} \begin{vmatrix} 11 & 2 & 1 \\ 13 & 3 & 1 \\ 12 & 1 & 4 \end{vmatrix} = \frac{1}{18} 18 = 1$$

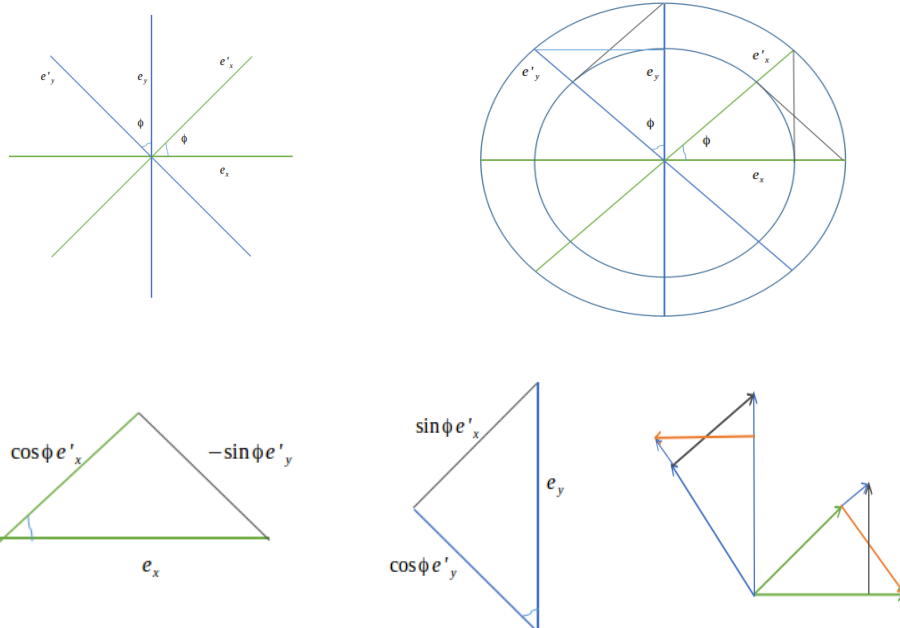
$$y = \frac{1}{18} \begin{vmatrix} 3 & 11 & 1 \\ 2 & 13 & 1 \\ 1 & 12 & 4 \end{vmatrix} = \frac{1}{18} 54 = 3$$

$$z = \frac{1}{18} \begin{vmatrix} 3 & 2 & 11 \\ 2 & 3 & 13 \\ 1 & 1 & 12 \end{vmatrix} = \frac{1}{18} 36 = 2$$

1.6 Coordinate transformation

Consider a rotation of the coordinate axes in 2D.

- We wish to find how the components A_x and A_y of a vector $\vec{A}$ in the unrotated system are related to A'_x and A'_y , its components in the rotated coordinate system.



- But it is easiest if we start with unit vectors, expressing the unit vectors, e_x and e_y in terms of the new ones, e'_x and e'_y

$$e_x = e_x \cos \phi + e_x \sin \phi$$

But $e_x \cos \phi = e'_x \cos \phi$ and $e_x \sin \phi$ fits with $-e'_y \sin \phi$

$$e_x = \cos \phi e'_x - \sin \phi e'_y$$

Similarly,

$$e_y = \sin \phi e'_x + \cos \phi e'_y$$

- In matrix multiplication formula

$$\begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e'_x \\ e'_y \end{pmatrix}$$

- An instructive way of writing the transformation is to rewrite those equations as

$$\begin{aligned} e_x &= (e_x \cdot e'_x) e'_x + (e_x \cdot e'_y) e'_y \\ e_y &= (e_y \cdot e'_x) e'_x + (e_y \cdot e'_y) e'_y \end{aligned}$$

- For a transformation matrix S' , $e = S' e'$

$$S' = \begin{pmatrix} e'_x \cdot e_x & e'_y \cdot e_x \\ e'_x \cdot e_y & e'_y \cdot e_y \end{pmatrix}$$

- Therefore for any vector $\vec{A}$ the transformation takes

$$A = S' A' \quad \text{or,}$$

$$A' = S A \quad \text{where,}$$

$$S' = \begin{pmatrix} e'_x \cdot e_x & e'_y \cdot e_x \\ e'_x \cdot e_y & e'_y \cdot e_y \end{pmatrix}, S = \begin{pmatrix} e'_x \cdot e_x & e'_x \cdot e_y \\ e'_y \cdot e_x & e'_y \cdot e_y \end{pmatrix}$$

- In 3D we have

$$S' = \begin{pmatrix} e'_x \cdot e_x & e'_y \cdot e_x & e'_z \cdot e_x \\ e'_x \cdot e_y & e'_y \cdot e_y & e'_z \cdot e_y \\ e'_x \cdot e_z & e'_y \cdot e_z & e'_z \cdot e_z \end{pmatrix}, S = \begin{pmatrix} e'_x \cdot e_x & e'_x \cdot e_y & e'_x \cdot e_z \\ e'_y \cdot e_x & e'_y \cdot e_y & e'_y \cdot e_z \\ e'_z \cdot e_x & e'_z \cdot e_y & e'_z \cdot e_z \end{pmatrix}$$

- Example: Consider three successive rotations in 3D.
 - The coordinates are rotated about the e_3 axis counterclockwise through an angle α in the range $0 \leq \alpha \leq 2\pi$
 - The coordinates are rotated about the e'_2 axis counterclockwise

through an angle β in the range $0 \leq \beta \leq \pi$

- The coordinates are now rotated about the e_3'' axis counterclockwise through an angle γ in the range $0 \leq \gamma \leq 2\pi$

- Calculate the transformation matrices and the overall transformation matrix

$$\begin{aligned}
 S_1 &= \begin{pmatrix} e'_x \cdot e_x & e'_x \cdot e_y & e'_x \cdot e_z \\ e'_y \cdot e_x & e'_y \cdot e_y & e'_y \cdot e_z \\ e'_z \cdot e_x & e'_z \cdot e_y & e'_z \cdot e_z \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 S_2 &= \begin{pmatrix} e''_x \cdot e'_x & e''_x \cdot e'_y & e''_x \cdot e'_z \\ e''_y \cdot e'_x & e''_y \cdot e'_y & e''_y \cdot e'_z \\ e''_z \cdot e'_x & e''_z \cdot e'_y & e''_z \cdot e'_z \end{pmatrix} = \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \\
 S_3 &= \begin{pmatrix} e'''_x \cdot e''_x & e'''_x \cdot e''_y & e'''_x \cdot e''_z \\ e'''_y \cdot e''_x & e'''_y \cdot e''_y & e'''_y \cdot e''_z \\ e'''_z \cdot e''_x & e'''_z \cdot e''_y & e'''_z \cdot e''_z \end{pmatrix} = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

1.7 The Eigenvalue Problem

Let $A = (a_{jk})$ be an $n \times n$ matrix and X a column vector. The equation $AX = \lambda X$, where λ is a number, is called an eigenvalue problem.

- Rewriting the eigenvalue problem as

$$(A - \lambda I)X = 0$$

- This equation will have a non-trivial solution only if

$$\det(A - \lambda I) = 0$$

- The solutions X of this equation are called eigenvectors and the corresponding values, λ are eigenvalues.

Example

Find the eigenvalues and normalized eigenvectors for the matrices.

$$A = \begin{pmatrix} -2 & \sqrt{5} \\ \sqrt{5} & -6 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Solution

$$\lambda_1 = -1, \quad X_1 = \begin{pmatrix} \sqrt{\frac{5}{6}} \\ \sqrt{\frac{1}{6}} \end{pmatrix}, \quad \lambda_2 = -7, \quad X_2 = \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ \sqrt{\frac{5}{6}} \end{pmatrix}$$